MULTIPLICITY OF THE TRIVIAL REPRESENTATION IN RANK-SELECTED HOMOLOGY OF THE PARTITION LATTICE

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Abstract. We study the multiplicity $b_S(n)$ of the trivial representation in the symmetric group representations β_S on the (top) homology of the rank-selected partition lattice Π_n^S. We break the possible rank sets S into three cases: (1) $1 \notin S$, (2) $S = 1, \dots, i$ for $i \ge 1$ and (3) $S = 1, \dots, i, j_1, \dots, j_l$ for $i, l \ge 1$, $j_1 > i + 1$. It was previously shown by Hanlon that $b_S(n) = 0$ for $S = 1, \dots, i$. We use a partitioning for $\Delta(\Pi_n)/S_n$ due to Hersh to confirm a conjecture of Sundaram [Su] that $b_S(n) > 0$ for $1 \notin S$. On the other hand, we use the spectral sequence of a filtered complex to show $b_S(n) = 0$ for $S = 1, \dots, i, j_1, \dots, j_l$ unless a certain type of chain of support S exists. The partitioning for $\Delta(\Pi_n)/S_n$ allows us then to show that a large class of rank sets $S = 1, \dots, i, j_1, \dots, j_l$ for which such a chain exists do satisfy $b_S(n) > 0$. We also generalize the partitioning for $\Delta(\Pi_n)/S_n$ to $\Delta(\Pi_n)/S_\lambda$; when $\lambda = (n-1, 1)$, this partitioning leads to a proof of a conjecture of Sundaram about $S_1 \times S_{n-1}$-representations on the homology of the partition lattice.

1. Introduction

The natural action of the symmetric group S_n on $\{1, \dots, n\}$ gives rise to a rank-preserving, order-preserving action on the lattice Π_n of partitions of $\{1, \dots, n\}$ ordered by refinement. The resulting S_n-action permuting chains of comparable poset elements yields an S_n-representation on the top homology of the order complex of the partition lattice. We study the multiplicity $b_S(n)$ of the trivial representation in the representation β_S of the symmetric group S_n on the homology of the partition lattice Π_n restricted to rank set S for various $S \subseteq [n-2]$. Questions about these multiplicities were first suggested in [St1] and studied quite extensively, using symmetric functions, in [Su].

We approach these questions from two other angles: spectral sequences of filtered complexes and partitioning of quotient complexes. A partitioning of the quotient complex $\Delta(\Pi_n)/S_n$ lends itself well to proving lower bounds on $b_S(n)$, while spectral sequences of filtered complexes seem well-suited to giving upper bounds. One of our interests is finding cases where we can get the two bounds to meet and seeing how the two very different methods make use of the same information. In particular, we give results about when $b_S(n)$ is positive and when it is 0 (as well as when a related multiplicity $b'_S(n)$ is positive), including proofs of two conjectures of Sundaram [Su].

Recall that the **order complex** $\Delta(P)$ of a finite poset P with minimal and maximal elements $\hat{0}$ and $\hat{1}$ is the simplicial complex comprised of an i-face for each chain $\hat{0} < u_0 < \cdots < u_i < \hat{1}$ of comparable poset elements. Whenever a group G acts on a graded poset P in a rank-preserving, order-preserving fashion, the group

This work was supported by an NSF-AWM Mentoring Travel Grant. The first author was also supported by NSF Grant DMS-9500979.

also permutes the poset chains, or equivalently the faces in its order complex. This G-action on $\Delta(P)$ commutes with the boundary map, so the action on chains also gives rise to a G-representation on each of the homology groups of $\Delta(P)$. The action on chains also may be restricted to any rank set S giving rise to a group representation α_S permuting the chains of support S and to representations on the homology of $\Delta(P)$ restricted to rank set S.

We will be interested in the alternating sum $\beta_S = \sum_{T \subseteq S} (-1)^{|S-T|} \alpha_T$ of S_n-representations α_T on chains. When P is a Cohen-Macaulay poset, then $\Delta(P)$ (and each of its rank-selected subcomplexes) only has top homology, in which case β_S is the G-representation on the top homology group in the rank-selected complex Δ^S obtained by restricting Δ to rank set S. The partition lattice is a Cohen-Macaulay poset, and we will be interested in the multiplicity $b_S(n)$ of the trivial representation in $\beta_S(n)$. Our results about when $b_S(n)$ is positive come out of an analysis of the flag h-vectors of the quotient complex $\Delta(\Pi_n)/S_n$, defined as follows:

Definition 1.1. *The* **flag f-vector** *of a $(d-1)$-dimensional balanced simplicial complex Δ is a vector with coordinates f_S for each subset $S \subseteq \{1, \ldots, d\}$ of the set of vertex colors for Δ where f_S counts how many faces in Δ have vertices colored exactly by S. The* **flag h-vector** *is the alternating sum $h_S = \sum_{T \subseteq S} (-1)^{|S|-|T|} f_T$, or alternatively, $h_S = (-1)^{|S|} \tilde{\chi}(\Delta^S)$. See [St2] for more background.*

In our case, the vertices of $\Delta(\Pi_n)$ are colored by poset rank, and $\Delta(\Pi_n)$ is balanced because no two elements of a chain have the same rank, implying no two vertices in the same face in $\Delta(\Pi_n)$ are assigned the same color.

The quotient complex $\Delta(P)/G$ consists of the G-orbits of faces in $\Delta(P)$, and it inherits the balancing by poset rank from $\Delta(P)$ when P is graded and G preserves rank. Note that $\Delta(P)/G$ typically is not the same as the order complex $\Delta(P/G)$ of the quotient poset, and in particular $\Delta(\Pi_n/S_n) \neq \Delta(\Pi_n)/S_n$; there are elements $u < v, u' < v' \in P$ such that $u = \sigma u', v = \tau v'$ for $\sigma, \tau \in G$ but $u < v$ is not in the same G-orbit as $u' < v'$. The quotient complex often is not a simplicial complex, but it is always a boolean cell complex, i.e. a regular cell complex in which each cell has the combinatorial type of a simplex.

The multiplicity $\langle \alpha_S, 1 \rangle$ of the trivial representation within the group action α_S on chains of support S equals the number of orbits in the action α_S, i.e. it equals $f_S(\Delta(P)/G)$. As observed in [Re], this implies that

$$\langle \beta_S, 1 \rangle = \left\langle \sum_{T \subseteq S} (-1)^{|S-T|} \alpha_T, 1 \right\rangle = \sum_{T \subseteq S} (-1)^{|S-T|} f_T(\Delta(P)/G) = h_S(\Delta(P)/G).$$

Hence, we will study flag h-vectors of quotient complexes as a way of getting at $b_S(n) = \langle \beta_S, 1 \rangle$. In particular, we will use the fact that when a balanced complex Δ is shellable or partitionable, then $h_S(\Delta)$ counts minimal faces of support S in the shelling or partitioning.

Definition 1.2. *A pure simplicial complex Δ is* **partitionable** *if the set of faces may be partitioned into a direct sum*

$$\Delta = [G_1, F_1] \cup \cdots \cup [G_k, F_k]$$

of intervals of boolean type where $F_1, \ldots, F_k$ are the facets of Δ and G_i is a face of F_i for $1 \leq i \leq k$. The complex Δ is **shellable** *if the facets may be ordered $F_1, \ldots, F_k$ so that for $2 \leq j \leq k$, the set $F_j \setminus \cup_{i<j} F_i$ of faces belonging to F_j but*

not to any earlier facet, has a unique minimal element G_j. Thus, a shelling may be viewed as a type of partitioning.

Further background may be found in [St2]. We will use a very complicated partitioning for $\Delta(\Pi_n)/S_n$ given in [He] to show $b_S(n) > 0$ for various classes of S by exhibiting facets F_i with minimal faces G_i of support S. The partitioning for $\Delta(\Pi_n)/S_n$ has the property that for a very large class of facets F_i, the minimal faces G_i may be described in terms of a generalized notion of ascents and descents in a chain-labeling on orbits of saturated chains in Π_n^*. Our strategy is to construct facets achieving various descent sets S to show that $b_S(n) > 0$ for these rank sets S.

Denote by Π_n^S the rank-selected subposet of the partition lattice consisting of those poset elements of rank r for some $r \in S \subseteq [n-2]$. We show in Section 3 that $\langle \beta_S, 1\rangle = 0$ for nearly all other S by using spectral sequences to prove that the trivial-isotypic piece of $H(\Pi_n^S)$ vanishes. The middle ground that is not covered by our results seems fairly subtle. Section 4 generalizes the partitioning of $\Delta(\Pi_n^*)/S_n$ to $\Delta(\Pi_n^*)/S_\lambda$ in order to prove a second conjecture of Sundaram, regarding representations of $S_1 \times S_{n-1}$ on homology.

2. Partitioning results

In Sections 2 and 4, we will study the flag h-vector for $\Delta(\Pi_n)/S_n$ and $\Delta(\Pi_n)/S_\lambda$, respectively, using partitionings which express flag h-vector coordinates in terms of ascents and descents in a chain-labeling for orbits of saturated chains in the dual poset; this virtually necessitates the use of ranks in the dual poset within these proofs, despite the fact that related results and conjectures of Sundaram and our own spectral sequence arguments are phrased in terms of the ranks of Π_n instead of Π_n^*. In an effort to minimize confusion in converting back and forth between the rank sets for the partition lattice and its dual, we will denote by S^* the rank set in Π_n^* which translates to rank set S in Π_n (or equivalently to corank set S^* in Π_n). For the sake of consistency, the statements of all results will be in terms of rank sets S; however, all of the proofs in Sections 2 and 4 (as well as Theorem 3.1) work internally with ascents and descents in the dual poset, so we systematically refer to rank sets S^* inside these proofs. Our arguments also may sometimes abuse notation by referring to chains when we always mean orbits of chains.

Let us depict the facets in $\Delta(\Pi_n^*)/S_n$ (namely the S_n-orbits of saturated chains in Π_n^*) by diagrams consisting of n balls with bars separating them (or arrows indicating where these bars are to be inserted) and the numbers from 1 to $n-1$ labeling the bars (or the arrows). The bar labels indicate the ranks in Π_n^* (or equivalently the coranks in Π_n) at which the bars are inserted in the course of progressively refining a single block of n objects into n singleton blocks. The balls represent the numbers $1, \ldots, n$ being partitioned, since we may freely permute these n numbers without switching orbit. Figure 1 gives an example (to be used again later) which begins by refining a block of size 10 into children of sizes 2,8 and next refines the block of size 8 into children of sizes 2,6.

Each refinement step takes an ordered partition (with the block ordering coming inductively from the choice of root for the chain orbit) and splits one of its blocks into two smaller blocks by inserting a bar into the block. Thus, we preserve the order of the original blocks and must only choose which of the two new blocks goes

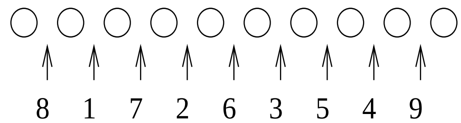

FIGURE 1. Facet achieving D^3D^4A

to the left of the other in the former position of the parent block. We make the convention of placing each bar as far to the left as possible among all choices that would give the same saturated chain orbit. In particular, placing a bar at position i in a block of size n is equivalent to placing the bar at position $n-i-1$ in that block, and we choose the position farther to the left. The other situation in which there is a choice to make is when there are equivalent blocks of the type to be split, in which case we refine the leftmost such block. This only happens if the blocks have the same size and were created from the same parent at the same step.

The partitioning in [He] uses a chain-labeling in which each covering relation $u \prec v$ is labeled by a triple (i, w, r) consisting of the position i of the bar being inserted, the word w recording the positions of all bars in v and the rank r at which the block being split at step $u \prec v$ was itself created. In [He], there is also some sorting of equivalent blocks just prior to the labeling of each covering relation, but we may safely ignore this because our arguments based on the partitioning will only consider facets in which no sorting takes place (or in a few cases ranks at which sorting does not occur within facets that do require sorting elsewhere). We say that a saturated chain orbit in Π_n^* has a **topological descent** at rank i if the pair of covering relations $u \prec v$ and $v \prec w$ labeled (i, w, r) and (i', w', r'), respectively, satisfy any of the following conditions:

(1) the bar inserted by $u \prec v$ is farther to the right than the bar insertion from $v \prec w$ (in which case $i > i'$, so the labels decrease);
(2) the bar insertions $u \prec v$ and $v \prec w$ proceed from left to right splitting a single block of u into three smaller blocks with the left child from the $u \prec v$ refinement strictly larger than the left child from the $v \prec w$ refinement step;
(3) the bar insertions $u \prec v$ and $v \prec w$ refine a single block of u into three children such that the left children resulting from the $u \prec v$ and $v \prec w$ covering relations both have size two and the latter gets refined to singletons before the former.

All other ranks in the saturated chain orbit are called (topological) ascents. When the above chain-labeling is used to lexicographically order facets, the topological descents are the ranks which may be omitted from a facet to obtain codimension one faces that also belong to lexicographically earlier facets. For most facets F_i, the ranks of the topological descents in Π_n^* are exactly the ranks included in the minimal face G_i in the partitioning for $\Delta(\Pi_n^*)/S_n$, so our aim will be to find facets achieving various topological descent sets.

To be more precise, the support of G_i is exactly the ranks of the topological descents in F_i if F_i satisfies the nontrivial, non-equal block condition, as stated just prior to Theorem 3.1. We should remark that facets F_i in $\Delta(\Pi_n^*)/S_n$ violating the nontrivial, non-equal block condition have minimal faces G_i whose support is not

exactly the set of topological descents in F_i, but our $b_S(n) > 0$ results only need to make use of facets which do satisfy the non-equal block condition; when we give partitioning proofs of $b_S(n) = 0$ and $b_S(n) = b_S(m)$ results, we must consider all facets, but then we use the fact that even for facets F_j violating the non-equal block condition, the ranks not involving equal blocks are in G_j if and only if they are topological descent ranks. These ranks which are governed by descents are enough to show that these facets do not have minimal faces of support S forbidden in the $b_S(n) = 0$ results and these ranks also suffice to set up the bijection needed in the new proof of Stanley's $b_S(n) = b_S(m)$ result.

Let us represent the rank set $S^* = \{i_1, i_2, \ldots, i_r\}$ by the word $w(S^*) \in \{A, D\}^{n-2}$ which has a D (for descent) at each position $i \in S^*$ and an A (for ascent) at each of the remaining position. This reflects the fact that in a lexicographic shelling a saturated chain having descents at exactly the positions in S^* would increase the value of the flag h-vector coordinate $h_S(\Delta(P))$ by one. Now let us turn to the following two conjectures of Sundaram [Su]:

(1) If $1 \notin S$, then $b_S(n) \neq 0$.
(2) Let $b'_S(n)$ be the multiplicity of the trivial representation in the representation of $S_{n-1} \times S_1$ on homology of the rank-selected partition lattice Π_n^S. If $S = \{1, \ldots, i\}$ then $b'_S(n) = 1$ and otherwise $b'_S(n) > 1$.

Sundaram proved in [Su] the first part of her second conjecture, namely that $b'_S(n) = 1$ for $S = \{1, \ldots, i\}$. Theorem 2.1 will confirm the first conjecture. We defer the proof of the second conjecture until Section 4 because it will rely on a partitioning for $\Delta(\Pi_n^*)/S_1 \times S_{n-1}$ that is given in that section. Theorem 2.1 is followed by a slight generalization, and then we give short new proofs based on partitioning for results of Sundaram [Su] and Stanley [St1].

Theorem 2.1. *If $1 \notin S$ then $b_S(n) > 0$.*

Proof. The requirement $1 \notin S$ translates to $n - 2 \notin S^*$. Let us exhibit facets in $\Delta(\Pi_n^*)/S_n$ achieving descent set S^* for each such pair (S, n). First we show how to achieve the word $w(S^*) = D^{n-3}A$ for any n, handling the cases of n even and odd separately. After this, we will show how to achieve concatenations of such words, so as to obtain any word $w(S^*)$ ending in an ascent.

When $n = 2k$, we achieve $D^{k-2}D^{k-1}A$ by first inserting $k - 1$ bars from left to right sequentially into even positions $2, 4, \ldots, 2k - 2$, then sequentially inserting bars from right to left into odd positions $2k - 3, 2k - 5, 2k - 7, \ldots, 1$; finally, we obtain an ascent by concluding with a bar in position $2k - 1$. Figure 1 gives an example for $k = 5$. Note that the first $k - 2$ pairs of consecutive bar insertions are topological descents because the codimension one face which skips from the bar insertion into position $2i$ directly to partition with additional bars at positions $2i + 2, 2i + 4$ also belongs to the lexicographically earlier facet which reverses the order (later in the chain) in which bars are inserted into positions $2i + 1$ and $2i + 3$. The next $k - 1$ pairs of consecutive bar insertions proceed from right to left and hence are also topological descents. The final pair of consecutive bar insertions into positions $1, 2k - 1$ gives a topological ascent.

Similarly, for $n = 2k+1$, insert bars sequentially into even positions $2, 4, \ldots, 2k-2$ then odd positions $2k-1, 2k-3, 2k-5, \ldots, 3, 1$ and finally into even position $2k$, as in Figure 2. The only change is to note that the pair of consecutive insertions

at positions $2k-2, 2k-1$ is a topological descent since there is a lexicographically earlier facet which instead inserts bars first into position $2k-3, 2k-1$ and shares a codimension one face skipping rank $k-1$ with our facet.

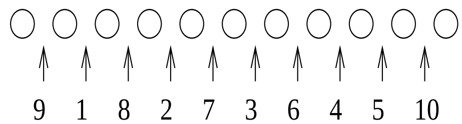

FIGURE 2. Facet achieving D^4D^4A

To achieve any sequence of ascents and descents that ends in an ascent, note that the words $D^{n_1}A$ and $D^{n_2}A$ may be concatenated as follows: apply the above

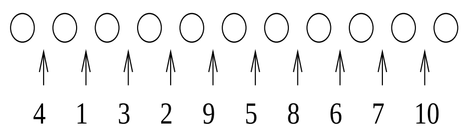

FIGURE 3. Facet achieving D^3AD^4A

construction for $D^{n_1}A$ placing bars into the leftmost n_1 available positions, then rather than concluding with a bar insertion into position n_1+1, instead place this bar in position n_1+2, creating the leftmost block of size 2 in the beginning of the construction for $D^{n_2}A$ using the remaining bar positions. If there are more than two words to concatenate, proceed greedily in this fashion from left to right among available bar positions. Figure 3 gives an example for D^3AD^4A. $\square$

Now let us strengthen Theorem 2.1. As before. we state the result in terms of rank set S in Π_n, but then use (orbits of) saturated chains in Π_n^* in the proof.

Theorem 2.2. *Let $S = \{1, \ldots, i, j_1, \ldots, j_l\}$ for $j_1 - i > 1$. If $i \leq l$, then $b_S(n) > 0$.*

PROOF. First let us consider the case when $i = |S|/2$. This means S^* has exactly i descents interspersed with ascents and then has at least one ascent immediately preceding the final string of i consecutive descents. To realize this pattern in a saturated chain in Π_n^*, begin by inserting bars from left to right creating blocks of size 1 or 2, making a block of size 2 at each descent and a block of size 1 for each ascent. This accounts for the first i descents and the ascents in which they are interspersed. Now we refine the remaining $i+1$ blocks of size 2 from right to left to achieve the last i descents. Notice that there must be one extra block of size 2 immediately to the right of the rightmost block of size one that we have created, because the size of n dictates that there must be $i+1$ further refinements, forcing this rightmost block to have size 2.

Now suppose $l > i$. Let us then break $w(S^*)$ into subwords w_1, w_2 such that $w(S^*) = w_1 \circ w_2$ and the word w_1 has exactly $l-i$ descents including a terminal one. If w_2 begins with an ascent, then we achieve w_1A by the construction for words ending in an ascent applied to the leftmost bar positions, and then we achieve w_2' such that $w_2 = Aw_2'$ by the construction of the previous paragraph on the

remaining bar positions. On the other hand, if w_2 begins with a descent, then we use the previous theorem's construction for the maximal word w_1' such that $w_1 = w_1' \circ D^{r_1}$, since w_1' must end in an ascent; then we achieve $D^{r_1+r_2}Aw_2''$ such that $w_2 = D^{r_2}Aw_2''$ by the following construction: if r_1 is even, then insert bars sequentially from left to right into even positions $2, 4, \ldots, 2r_2 + r_1$, then from right to left into the odd positions $2r_2+r_1-1, 2r_2+r_1-3, \ldots, 2r_2+r_1-(r_1-1)$, then use the $i = |S|/2$ procedure for the bar positions to the right of $2r_2 + r_1$ and conclude by placing bars right to left into consecutive odd positions $2r_2+r_1-(r_1+1), \ldots, 1$; for r_1 odd, sequentially insert bars into even positions $2, 4, \ldots, 2r_2 + r_1 - 1$, then insert bars right to left in odd position $2r_2+r_1, 2r_2+r_1-2, \ldots, 2r_2+1$, and finally use the $i = |S|/2$ procedure on the positions to the right of $2r_2 + r_1$. ☐

Sundaram [Su, p. 288] showed that $b_S(n) = 0$ whenever any of the following conditions are met:

(1) $S = \{1, \ldots, i\}$ with $i > 0$. This result of [Ha] and [Su] is recovered by partitioning in [He].
(2) $[1, \lfloor (n+1)/2 \rfloor] \subseteq S$.
(3) $S = [1, r] \cup a$ for $a \notin [\binom{r+2}{2}, n - r - 1]$.
(4) $S = [1, r] - k$ for n even and $k = n/2 - 1$, provided $k = \frac{n}{2} - 1 \leq r \leq n - 4$.

Sundaram asked (private communication) if these results could easily be recovered using the partitioning for $\Delta(\Pi_n)/S_n$. We now give proofs by partitioning for these results and then provide further results about when $b_S(n) > 0$ and when $b_S(n) = 0$. Let us begin with Item 2.

Theorem 2.3. *If $[1, \lfloor (n+1)/2 \rfloor] \subseteq S$, then $b_S(n) = 0$.*

Proof. Consider the partition immediately before the final string of descents in the orbit of any saturated chain in Π_n^*. At this point, less than half the bars have been inserted, so some blocks have size larger than 2. Each such block forces an ascent, a contradiction. ☐

Next we give a slight strengthening of Sundaram's fourth result.

Theorem 2.4. *If $S = [1, r] - k$ for $k > r/2$, then $b_S(n) = 0$.*

Proof. Suppose the orbit $O(C^*)$ of some saturated chain in Π_n^* achieves the set of coranks S^* for $S = [1, r] - k$, i.e. suppose it achieves $w(S^*) = A^{n-2-r}D^{r-k}AD^{k-1}$. For $O(C^*)$ to begin with $n - 2 - 4$ ascents, bars must be inserted left to right creating blocks of nondecreasing size. Each of these blocks of size larger than 2 will necessitate an ascent to complete its refinement some time after the first descent. Thus, there may be at most one block of size larger than 2 created by the initial string of ascents. In addition, any blocks of size 2 which are created initially must later be split from left to right for none of the initial bar insertions to be topological descents.

Thus, the initial string of ascents creates at most one block of size 2, so it creates some number of trivial blocks, followed by at most one block of size 2, and then at most one larger block. The first topological descent must come from proceeding right to left in order to refine the unique block of size two or else we get a topological descent by splitting off the leftmost singleton in a block of size larger than two. In

either case, this step must be followed by an ascent. Now we need to achieve a nonempty string of descents to complete the refinement, but it is impossible to completely refine the block which had size larger than two using only topological descents, by the same argument that was used to show $b_{1,\dots,i}(n) = 0$ in [He]. □

Another application of the partitioning is a simple proof for the following result of Stanley [St1].

Theorem 2.5. *Let $S = \{j_1, \dots, j_l\}$. Then $b_S(m) = b_S(n)$ if $m, n > 2j_l$.*

Proof. A saturated chain in Π_n^* achieving a set S as above must begin with $n - j_l - 2$ consecutive ascents, so we must insert bars left to right creating blocks of nondecreasing size. These ascents create at most j_l blocks of size greater than one since there are only j_l remaining refinement steps to completely refine these blocks. Thus, the initial ascents must create only blocks of size one from left to right for the first $n - 2j_l - 2$ bar insertions. We get a bijection between facets in $\Delta(\Pi_n^*)/S_n$ and in $\Delta(\Pi_m^*)/S_m$ which contribute minimal faces of support S to their respective partitionings, by changing the number of initial blocks of size one and otherwise letting the bar insertions agree once we have split off the necessary number of singletons from each facet. □

3. Conditions under which $b_S(n) = 0$

In this section, we will recover Sundaram's third result using spectral sequences, and then we generalize her result by replacing the single rank a by a collection of ranks which are disjoint from the consecutive initial ranks $1, \dots, i$. First we use a partitioning for $\Delta(\Pi_n^*)/S_n$ to obtain conditions under which $b_S(n) > 0$, before using spectral sequences to show $b_S(n) = 0$ for nearly all other S.

Let $S = \{1, \dots, i, j_1, \dots, j_l\}$ with $j_1 - i > 1$, so $S^* = \{n-1-j_l, n-1-j_{l-1}, \dots, n-1-j_1\} \cup [n-1-i, n-2]$. Let $E(C^*)$ denote the lexicographically smallest extension of a face C^* in $\Delta(\Pi_n^*)/S_n$ to a saturated chain orbit, based on the chain-labeling of [He]. Let us say that a chain $C = \alpha_1 \prec \cdots \prec \alpha_i < \beta_1 < \cdots < \beta_l$ of support $\{1, \dots, i, j_1, \dots, j_l\}$ satisfies the **non-equal block condition** if the extension $E(C^*)$ to a saturated chain does not have any pairs of equal blocks created from the same parent either in a single refinement step or in consecutive refinement steps. If we relax this requirement to allow equal blocks of size two, we call this the **nontrivial, non-equal block condition**. Theorem 3.1 is not tight in that there are rank sets $S = \{1, \dots, i, j_1, \dots, j_l\}$ such that $b_S(n) > 0$ but where every facet whose minimal face has support S violates the non-equal block condition; the situation seems much more subtle when one removes the non-equal block condition. Let $Stab(C)$ denote the stabilizer of a chain C.

Theorem 3.1. *Let $C = \alpha_1 \prec \cdots \prec \alpha_i < \beta_1 < \cdots < \beta_l$ be a chain in Π_n of support $S = \{1, \dots, i, j_1, \dots, j_l\}$ for $j_1 > i + 1$ that satisfies the non-equal block condition. Furthermore, suppose that α_i has i nontrivial blocks $B_1, \dots, B_i$ of size 2 and that β_1 has $i+1$ nontrivial blocks $C_1, \dots, C_{i+1}$ belonging to distinct $Stab(\beta_1 < \cdots < \beta_l)$-orbits such that $B_r \subseteq C_r$ for $1 \le r \le i$. Then $b_S(n) > 0$.*

Proof. Let C be a chain as above. Either $E(C^*)$ will have (topological) descents at exactly the ranks in S^* or we will construct a closely related chain C' with the desired topological descent set for $E(C')^*$ such that C' satisfies the non-equal block

condition. Thus, either $E(C^*)$ or $E((C')^*)$ will contribute to the partitioning a minimal face of support S^*, implying $b_S(n) > 0$.

The saturated chain $E(C^*)$ is obtained by extending each interval $u < v$ in C^* by inserting bars left to right and splitting each block of u from left to right into nondecreasing pieces, so that each rank in the extension is a topological ascent. Thus, the topological descents of $E(C^*)$ are a subset of the ranks in C^*, i.e. they are a subset of S^*. Furthermore, $E(C^*)$ must have topological descents at the topmost i ranks because C^* may be chosen to conclude by refining from right to left the $i+1$ blocks of size 2 which are children of the $i+1$ blocks in β_1 that are in distinct $Stab(\beta_1 < \cdots < \beta_l)$ orbits. As an example, the chain orbit $C^* = (00|000|00000 < 00|0|00|0|0|0|0|0 < 00|0|0|0|0|0|0|0|0 < \hat{1})$ of support $2,7,8$ extends to a chain $E(C^*)$ which sequentially inserts bars in positions $2,5,3,6,7,8,9,4,1$, and this has descents at ranks $2,7,8$; it contributes to flag h-vector coordinate $h_{2,7,8}$ for $\Delta(\Pi_{10}^*)/S_{10}$ or equivalently to $h_{9-2,9-7,9-8} = h_{1,2,7}$ for $\Delta(\Pi_{10})/S_{10}$.

Let us next consider the lower ranks in $E(C^*)$ and specifically how to turn ascents at ranks in C^* into topological descents. If there is an ascent at a rank $u \in E(C^*)$, then the bar inserted at the covering relation $t \prec u$ in $E(C^*)$ is to the left of the bar inserted at the covering relation $u \prec v$ in $E(C^*)$, and the two bars are either (1) inserted in different blocks or (2) inserted into a single block creating children B, B', B'' from left to right such $|B| \le |B'|$.

For the moment, let us assume that t and v are not in C^*, so that our goal will be to replace the ascent at u by a descent while preserving the ascents t and v. For ascents of type (1), we obtain from C^* a new chain $(C')^*$ by inserting the right bar before the left one. Note that the necessary ascents are preserved since the label leading upward to u was increased in value and the label leading upward from u was decreased; furthermore, nonequivalent blocks in C^* are still nonequivalent in $(C')^*$, preserving the requirement about $Stab(\beta_1 < \cdots < \beta_l)$ orbits. For ascents of type (2), we replace the consecutive bar insertions which give left children B, B' such that $|B| \le |B'|$ by left to right bar insertions instead sequentially yielding left children B', B to produce a topological descent. In the remainder of $(C')^*$, we refine B, B' just as C^* would, though this could in theory impact later ascents and descents at ranks in C^* since now B' is to the left of B. We can choose our chain to avoid turning ascents to descents and vice-versa as long as we proceed from lower to higher ranks in creating C'. This modification of C^* into $(C')^*$ for type 2 also gives ascents immediately above and below the descent at u, because in order for the bar creating B to be farther to the right than the bar inserted just before it, the bar which instead creates B' must also be farther to the right, and since $|B'| > |B|$, we observe that B' is also larger than any block B'' created just prior to B from the same parent as B, since we would have $|B''| < |B|$. Similarly, we are assured of an ascent immediately after the descent at u, and again we have preserved block-nonequivalence as needed. Also, the left child is never larger than the right child in a refinement since $|B'| \le |B''|$ and $|B| \le |B''|$.

Now let us more generally consider the possibility that the ascent u is among r consecutive elements $u_1, \ldots, u_r \in C^*$ for $r \ge 1$. Let us describe how to obtain $(C')^*$ which has topological descents at all of these r consecutive ranks and ascents immediately above and below them. Let $u_0 \prec u_1$ and $u_r \prec u_{r+1}$ be the covering relations of $E(C^*)$ immediately below and above the r consecutive ranks. In C', we refine the set of blocks in u_0 from right to left and within each block of u_0 insert

bars left to right with new blocks decreasing in size from left to right. Thus we get a string of descents. This is immediately preceded and followed by ascents, since (just as in the $r = 1$ case) the label for $u_1 \prec u_2'$ in $E(C')^*$ is smaller than the corresponding label in $E(C^*)$, and the label for $u_{r-1}' \prec u_r$ in $E(C')^*$ is larger than the corresponding label in $E(C^*)$. Our conversion of C to C' also preserves block-nonequivalence as needed. □

Next we use a spectral sequence for a filtered complex to give upper bounds on $b_S(n) = \langle 1, \beta_S(\Pi_n)\rangle$ by showing that the trivial-isotypic piece of $E^2(\Delta(\Pi_n^*))$ vanishes except when a certain type of chain of support S exists. See [We] for background on spectral sequences and in particular on how they give (upper) approximations on homology. We begin with a new proof that $b_{\{1,\dots,i\}}(n)$ which we then generalize from rank set $[1,r]\cup\{a\}$ to rank set $[1,r]\cup\{a_1\cup\cdots\cup a_l\}$ for $l \geq 1$. Now we come to Sundaram's third result.

Theorem 3.2. *Let $S = [1,r]\cup a$ for $a \notin [\binom{r+2}{2}, n-r-1]$. Then $b_S(n) = 0$.*

Proof. Consider the rank selection $\{1,2,\dots,i,j\}$ for any $i < j < \binom{i+2}{2}$. Let C be the set of chains in Π_n which are supported on a subset of these ranks.

For each chain Γ in C, define $f(\Gamma)$ to be $2J(\Gamma)+I(\Gamma)$ where $J(\Gamma) = 1$ if Γ contains an element of rank j, and $J(\Gamma) = 0$ otherwise; $I(\Gamma)$ is defined to be the rank of the maximal element of Γ which has rank at most i.

Note that $\partial(\Gamma)$ is a linear combination of chains Γ' with $f(\Gamma') \leq f(\Gamma)$. So f is a filtering on the complex (C,∂) and so we can approximate the homology $H(C,\partial)$ with $E^1 = H(C,\partial^0)$ where ∂^0 is the piece of the boundary ∂ which is fixed by the filtration function.

It is easy to see what ∂^0 does: if Γ has the form

$$\Gamma = \hat{0} < x_1 < \cdots < x_s < y < \hat{1}$$

where y is of rank j, then

$$\partial^0(\Gamma) = \sum_{l=1}^{s-1}(-1)^{l-1}\Big\{\hat{0} < \cdots < \hat{x_l} < \cdots < x_s < y < \hat{1}\Big\}.$$

If Γ is of the form $\hat{0} < x_1 < \cdots < x_s < \hat{1}$ where $\mathrm{rk(x_s)} \leq \mathrm{i}$, then

$$\partial^0(\Gamma) = \sum_{l=1}^{s-1}(-1)^{l-1}\Big\{\hat{0} < \cdots < \hat{x_l} < \cdots < x_s < \hat{1}\Big\}.$$

By examining the form of ∂^0, one sees that (C,∂^0) can be split as a direct sum

$$(C,\partial^0) = \bigoplus_{\alpha\in R^{\leq i}} C([\hat{0},\alpha],\delta) \oplus \bigoplus_{\substack{\gamma\in R^{\leq i}\\ \beta\in R^j,\gamma<\beta}} C([\hat{0},\gamma],\delta). \tag{3.1}$$

Here $C([\hat{0},\alpha],\delta)$ denotes the usual order complex of the poset $[\hat{0},\alpha]$. Let $R^{\leq i}$ denote the set of poset elements of rank at most i, and let R^j denote the set of poset elements of rank exactly j.

Note that $f(\sigma\Gamma) = f(\Gamma)$ for $\sigma \in S_n$. Therefore S_n commutes with the boundary ∂^0 and so it makes sense to talk about the S_n-module structure of the complex (C,∂^0). The S_n-module structure is best described in pieces corresponding to the two major summands in 3.1.

The summand $\bigoplus_{\alpha\in R^{\leq i}} C([\hat{0},\alpha],\delta)$ corresponds to the space of all chains which have no element of rank j. Likewise, the summand $\bigoplus_{\substack{\alpha\in R^{\leq i}\\ \beta\in R^{j},\alpha<\beta}} C([\hat{0},\alpha],\delta)$ corresponds to the span of all chains which do have an element of rank j. These two subspaces are S_n-invariant.

For the first summand, let $I=\{\alpha_1,\ldots,\alpha_l\}$ be the set of representatives from the orbits of S_n acting on $R^{\leq i}$. Then as an S_n-module, the first summand is

$$\bigoplus_{\alpha\in I} ind_{Stab(\alpha)}^{S_n}(C([\hat{0},\alpha],\delta)) \tag{3.2}$$

where $Stab(\alpha)$ denotes the stabilizer of α in S_n.

For the second summand we have a similar description. Let $J=\{\gamma_1<\beta_1,\gamma_2<\beta_2,\ldots,\gamma_m<\beta_m\}$ be a set of representatives from the orbits of S_n acting on the set $\{\gamma<\beta:\mathrm{rk}(\gamma)\leq \mathrm{i},\mathrm{rk}(\beta)=\mathrm{j},\gamma<\beta\}$. Then the second summand is

$$\bigoplus_{\{\gamma<\beta\}\in J} ind_{Stab(\gamma<\beta)}^{S_n}(C([\hat{0},\gamma],\delta)). \tag{3.3}$$

So, we can write the following expression for $E^1=H(C,\partial^0)$.

$$E^1=\bigoplus_{\alpha\in I} ind_{Stab(\alpha)}^{S_n}(H([\hat{0},\alpha]))\oplus\bigoplus_{(\gamma<\beta)\in J} ind_{Stab(\gamma<\beta)}^{S_n}(H([\hat{0},\gamma])). \tag{3.4}$$

Our goal is to show that the multiplicity of the trivial representation in E^∞ is 0. We will begin by characterizing the trivial-isotypic component in E^1.

As a notational convention, whenever V is a G-module for any group G, let V^G denote the trivial-isotypic component of V. By Frobenius Reciprocity,

$$(E^1)^{S_n}\approx\bigoplus_{\alpha\in I} H([\hat{0},\alpha])^{Stab(\alpha)}\oplus\bigoplus_{(\gamma<\beta)\in J} H([\hat{0},\gamma])^{Stab(\gamma<\beta)}. \tag{3.5}$$

Let $\alpha=A_1|\cdots|A_k|B_1|\cdots|B_l|C_1|\cdots|C_m$ where the A_u all have size 1, the B_v all have size 2 and the C_w all have size greater than 2. Corresponding to this decomposition,

$$St(\alpha)=\prod_{u\geq 1}(S_{m_u}\wr S_u) \tag{3.6}$$

where m_u is the number of blocks of α of size u and $S_{m_u}\wr S_u$ denotes a wreath product of symmetric groups. Likewise,

$$H([\hat{0},\alpha])\cong\bigotimes_{u\geq 3} H(\Pi_u)^{\otimes m_u}. \tag{3.7}$$

The action of $Stab(\alpha)$ on $H([\hat{0},\alpha])$ is given by an action of each $S_{m_u}\wr S_u$ on the tensor factor $H(\Pi_u)^{\bigotimes\, m_u}$. This wreath product action is the one in which the m_u copies of S_u act on $H(\Pi_u)$ in the usual way. The overlying copy of S_{m_u} acts according to the trivial representation if u is odd and the sign representation if u is even.

From this description of the action of $Stab(\alpha)$ on $H([\hat{0},\alpha])$, we will deduce that

$$H([\hat{0},\alpha])^{Stab(\alpha)}=\mathbb{C} \tag{3.8}$$

if α has a single block of size 2 and all other blocks of size 1, and that $H[\hat{0},\alpha])^{Stab(\alpha)}$ is 0 otherwise. To obtain 3.8, we use the well-known fact that

$$H(\Pi_u)^{S_u}=\begin{cases} 0 & \text{for}\quad \mathrm{u}\geq 3\\ \mathbb{C} & \text{for}\quad \mathrm{u}=1,2\end{cases} \tag{3.9}$$

Notice that the trivial representation of $Stab(\alpha)$ is $\bigotimes_{u\geq 1} 1_{m_u} \wr 1_u$, where 1_u denotes the trivial representation of S_u. Computing inner products, we see that a $Stab(\alpha)$-representation $\chi \wr \psi$ will not contain the trivial representation unless both χ and ψ do as well (cf. [JK, chapter 4]). Thus, 3.8 follows from 3.9 along with our above description of the action of $Stab(\alpha)$ on $H([\hat{0},\alpha])$.

Similar reasoning allows us to analyze the summand $\bigoplus_{(\gamma,\beta)\in J} H([\hat{0},\gamma])^{Stab(\gamma<\beta)}$ from (3.5). However, there is a subtlety here in that $Stab(\gamma<\beta)$ is not necessarily the full automorphism group of γ. For each block B of γ, the automorphism group $Stab(\gamma<\beta)$ will certainly contain S_B. However, different blocks of γ of the same size may not be interchanged by $Stab(\gamma<\beta)$ because they reside in blocks of β which have different size. The conclusion is that $H([\hat{0},\gamma])^{Stab(\gamma<\beta)} = 0$ unless every nontrivial block of γ has size 2 and if U, V are blocks of γ having size 2, then U and V are contained in blocks of β which have different sizes.

Let $\gamma < \beta$ be a pair for which $H([\hat{0},\gamma])^{Stab(\gamma<\beta)}$ is nonzero. We will identify the structure of $H([\hat{0},\gamma])^{Stab(\gamma<\beta)}$ more explicitly. Let $U_1,\ldots,U_t$ be the nontrivial blocks of γ (all of which have size 2). For each l, let V_l be the block of β which contains U_l. We know that $|V_l| = |V_m|$ implies $l = m$.

The poset $[\hat{0},\gamma]$ is isomorphic to the Boolean algebra B_t and so we know that the module $H([\hat{0},\gamma])$ has dimension 1 in degree t and dimension 0 in all other degrees. In addition we can explicitly give the homology representative $\rho[U_1,\ldots,U_t]$ in degree t:

$$\rho[U_1,\ldots,U_t] = \sum_{\sigma\in S_t} sgn(\sigma)\{\hat{0} < U_{\sigma 1} < U_{\sigma 2}|U_{\sigma 3} < \cdots < U_{\sigma 1}|U_{\sigma 2}|\cdots|U_{\sigma(t-1)} < \gamma\}. \tag{3.12}$$

Consider now the next step in the spectral sequence. We are going to compute $E^2 = H(E^1,\partial^1)$ where ∂^1 is the differential induced on E^1 by the piece of the original boundary which reduces the filtration function by 1. By the definition of f, $f(\Gamma') = f(\Gamma) - 1$ for chains $\Gamma' \subseteq \Gamma$ iff Γ' is obtained from Γ by removing the maximal element of Γ whose rank is in $\{1,\ldots,i\}$.

Referring to (3.8) we see that the S_n-invariants in the first summand of (3.5) consists of the single vector

$$\bigoplus_{type(\alpha)=2,1^{n-2}} H([\hat{0},\alpha]) = \langle \sum_{type(\alpha)=2,1^{n-2}} (\hat{0}<\alpha<\hat{1})\rangle_{\mathbb{C}}. \tag{3.13}$$

Applying ∂^1 to (3.13) gives a non-zero multiple of $(\hat{0}<\hat{1})$, hence the first summand contributes nothing to the kernel of ∂^1 so nothing to E^2.

Moving to the second summation, let $\gamma<\beta$ be a pair for which $H([\hat{0},\gamma])^{Stab(\gamma<\beta)}$ is nonzero. As with earlier notation, let γ have nontrivial blocks $U_1,\ldots,U_t$. It is straightforward to deduce from equation (3.12) that

$$\partial^1\rho[U_1,\ldots,U_t] = \sum_{i=1}^{t} (-1)^i \rho[U_1,\ldots,\hat{U}_i,\ldots,U_t]. \tag{3.14}$$

Let $V_1^{(1)}, V_2^{(1)},\ldots,V_{m_1}^{(1)}, V_1^{(2)}, V_2^{(2)},\ldots,V_{m_2}^{(2)},\ldots,V_1^{(s)},\ldots,V_{m_s}^{(s)}$ be the nontrivial blocks of β indexed so that $|V_i^{(j)}| = v_j$ for all i,j and with $2\leq v_1<v_2<\cdots<v_s$. Note that $\mathrm{rk}(\beta) = \sum_{\mathrm{l}=1}^{\mathrm{s}} \mathrm{m_l}(\mathrm{v_l}-1)$. Since $\mathrm{rk}(\beta) < \binom{\mathrm{i}+1}{2} = 1+2+\cdots+(\mathrm{i}+1)$, it follows that $s\leq i$. So there are at most i different non-trivial block sizes in β.

We now compute the contribution to $E^2 = H(E^1, \partial^1)$ made by the second summation. By (3.14), for each β at rank j, ∂^1 preserves $\bigoplus_{\gamma<\beta} H([\hat{0},\gamma])^{Stab(\gamma<\beta)}$.

Let $z_l = \binom{v_l}{2} m_l$ so that z_l is the number of pairs of numbers which occur in the same block of size v_l in β. Let $Z = Z_1 \cup Z_2 \cup \cdots \cup Z_s$ be a vertex set where Z_l contains z_l nodes $Z_l = \{x_1^{(l)}, \ldots, x_{z_l}^{(l)}\}$.

Define a simplicial complex Δ_β with vertex set Z by saying that Δ_β contains all subsets $S = \{s_1, \ldots, s_t\}$ such that $|S \cap Z_l| \leq 1$ for all l.

From the description of $H([\hat{0},\gamma])^{Stab(\gamma<\beta)}$ in (3.12) and the formula for the boundary ∂^1 it is clear that $H(\bigoplus_\gamma H([\hat{0},\gamma])^{Stab(\gamma<\beta)}, \partial^1) = H(\Delta_\beta)$ where $H(\Delta)$ is the ordinary simplicial homology of Δ.

Using arguments similar to previous ones,

$$(E^2)^{S_n} = \bigoplus_{\beta\in L} H(\bigoplus_{\gamma} H([\hat{0},\gamma])^{Stab(\gamma<\beta)}, \partial^1)^{Stab(\beta)} = \bigoplus_{\beta\in L} H(\Delta_\beta)^{Sym(Z_1)\times\cdots\times Sym(Z_s)}$$

where L is a complete set of representatives for the orbits of S_n on rank j and where $Z_1, \ldots, Z_s$ in the summand β are as above. But clearly $H(\Delta_\beta)^{Sym(Z_1)\times\cdots\times Sym(Z_s)} = 0$ as the projection by the trivial character of $Sym(Z_1)\times\cdots\times Sym(Z_s)$ maps Δ_β to the simplicial complex of all subsets of $\{1, 2, \ldots, s\}$ which is acyclic, because $s \leq i$.

A slight modification of this argument shows that the rank selected homology is 0 for ranks $1, 2, \ldots, i, j$ when $j \geq n - i$. The idea is that β has at most $n - j \leq i$ distinct blocks since there are only $n - j - 1$ covering relations from β to $\hat{1}$ in which to merge the blocks of β. □

Next we will extend the above argument, beginning with the choice of filter. If we allow $\Gamma = \alpha_1 \prec \alpha_2 \prec \cdots \prec \alpha_k < \beta_1 < \cdots < \beta_l$ above the α chain, then we let $J(\Gamma) = 2l$ and use the appropriately adjusted stabilizers and $\Delta_{\beta_1<\cdots<\beta_l}$.

Theorem 3.3. *If $b_S(n) > 0$ for $S = [1, i] \cup \{j_1, j_2, \ldots, j_l\}$ with $j_1 > i + 1$, then there exists a chain $\alpha_1 \prec \cdots \prec \alpha_i < \beta_1 < \cdots < \beta_l$ of support S such that (1) α_i consists of i blocks of size 2, and (2) β_1 includes nontrivial blocks $B_1, \ldots, B_{i+1}$ all belonging to distinct $Stab(\beta_1 < \cdots < \beta_l)$-orbits.*

PROOF. We will mimic the reasoning given in the case $\ell = 1$. First, we define the filtering f. Let

$$\Gamma = \alpha_1 < \alpha_2 < \cdots < \alpha_s < \beta_1 < \cdots < \beta_t$$

be a chain in the rank selection where the ranks of the α_u are in $[1, i]$ and the ranks of the β_v are in $\underline{j} = \{j_1, j_2, \ldots, j_\ell\}$. Define $f(\Gamma) = 2t + \mathrm{rk}(\alpha_\mathrm{s})$.

As in the proof of Theorem 3.2, the E^1 term in the spectral sequence corresponding to this filtration is a direct sum, over S_n-orbits of elements α_s with ranks in $[1, i]$, and over chains $\beta_1 < \cdots < \beta_t$ with ranks contained in the set $\underline{j}$ which also satisfy $\alpha_s < \beta_1$. The summands take the form

$$ind^{S_n}_{Stab(\alpha_s,\underline{\beta})}(H(\alpha_s)) \tag{3.15}$$

Here $Stab(\alpha_s, \underline{(\beta)})$ denotes the stabilizer of the chain $\alpha_s < \beta_1 < \cdots < \beta_t$.

We next compute the multiplicity of the trivial character in each of the summands in (3.15). As before, we invoke Frobenius reciprocity to argue that the multiplicity of the trivial character in (3.15) is equal to the multiplicty of the trivial character of $Stab(\alpha_s, \underline{\beta})$ in (3.15).

By reasoning similar to that used in the proof of Theorem 3.2, the trivial character has multiplicity 0 in (3.15) unless every nontrivial block of α_s has cardinality two. Assume therefore that every nontrivial block of α_s has cardinality two. If any pair of these nontrivial blocks are in the same orbit of $Stab(\alpha_s, \underline{\beta})$ then again the multiplicity of the trivial representation is 0.

Now consider the summand given by a chain $\underline{\beta} = \beta_1 < \cdots < \beta_t$ in the trivial-isotypic piece of the E^2 term of the spectral sequence. Similarly to when $l = 1$, this summand is isomorphic to the simplicial homology of a simplicial complex $\Delta_{\underline{beta}}$ whose ground set consists of the $Stab(\beta_1 < \cdots < \beta_t)$-orbits of the size 2 blocks that are contained in blocks of β_1. Let j be the size of this ground set. The faces of $\Delta_{\underline{\beta}}$ are the $Stab(\beta_1 < \cdots < \beta_1)$-orbits of chain elements $\alpha < \beta_1$ of rank at most i whose nontrivial blocks all have size 2. Thus, we are taking the simplicial homology of the $(i-1)$-skeleton of a $(j-1)$-simplex, so this is 0 unless $j > i$, as desired. Hence, $(E^2)^{S_n}$ vanishes unless there is a chain $\underline{\beta}$ with $j > i$. This implies $b_S(n) = 0$ unless there is such a chain, as desired. □

4. Partitioning $\Delta(\Pi_n)/S_\lambda$ and a conjecture of Sundaram

Next we prove the second conjecture of Sundaram [Su], by first giving a partitioning for $\Delta(\Pi_n)/S_1 \times S_{n-1}$, and more generally for $\Delta(\Pi_n)/S_\lambda$ for any Young subgroup of S_n. Instead of using bars to partition n balls, now we partition the multiset $\{1^{\lambda_1}, \ldots, k^{\lambda_k}\}$. Recall that [He] always chose the leftmost of equivalent positions in which to insert bars, splitting a block by inserting a bar with the smaller resulting block to its left; we more generally need an ordering on blocks which are subsets of $\{1^{\lambda_1}, \ldots, k^{\lambda_k}\}$ to decide which block goes to the left of each bar insertion. The entire partitioning argument of [He] will go through directly if we use any block ordering that satisfies the lengthening condition, defined as follows:

Definition 4.1. *A block order satisfies the* **lengthening condition (LC)** *if*

$$B \le B' \Rightarrow B \le BB',$$

where BB' denotes the concatenation of the two blocks, so the multiplicity in BB' of any letter appearing in both B and B' is the sum of the multiplicities.

Denote by w_B the word obtained from a block B by rearranging the letters in B into increasing order. If one views blocks as monomials, then any monomial term order will satisfy the lengthening condition. However, the distinguished block order (described below) satisfies the lengthening condition but is not a monomial term order; we will use the length-lex order to partition $\Delta(\Pi_n)/S_\lambda$ for an arbitrary λ.

Definition 4.2. *In the* **length-lex** *block order, a block B_1 is smaller than a block B_2 if $|B_1| < |B_2|$ or if $|B_1| = |B_2|$ and w_{B_1} is lexicographically smaller than w_{B_2}.*

When $\lambda_k = 1$, the distinguished block order will give a different partitioning that is more convenient for counting minimal faces of particular supports in the partitioning and in particular for proving a second conjecture of Sundaram.

Definition 4.3. *Suppose λ is a partition in which $\lambda_k = 1$, and let s be a letter appearing with multiplicity one. In the* **distinguished block order** *for $\Delta(\Pi_n)/S_\lambda$, a pair of blocks B_1, B_2 satisfy $B_1 < B_2$ if $s \in B_1, s \notin B_2$ or if $s \notin B_1, B_2$ and $B_1 < B_2$ in the length-lex block order. The letter s cannot belong to two different blocks, so we never need to compare blocks B_1, B_2 such that $s \in B_1, B_2$.*

It is not hard to check that both of the above block orders satisfy the lengthening condition. The lengthening condition and these two block orders were introduced in [HK].

Theorem 4.1. *Any block order satisfying the lengthening condition yields a partitioning for $\Delta(\Pi_n^*)/S_\lambda$. Hence, $\Delta(\Pi_n)/S_\lambda$ is partitionable using the length-lex order.*

Proof. Let us modify the chain-labeling for $\Delta(\Pi_n^*)/S_n$ as follows: label covering relations with ordered 4-tuples (i, w_B, W, r) where i is the number of bars to the left of the bar being inserted, w_B is the content of the block immediately to the left of this bar, W is the word obtained by concatenating all the block words to the left of the new bar and interspersing bar symbols between the block words, and let r be the rank at which the parent block P into which the bar is inserted was itself created. Precedence in the 4-tuple proceeds from left to right. The words w_B are ordered by a block order satisfying the lengthening condition. The words W are ordered by considering the first block where two words differ and then using our block order to compare these blocks. In the partitioning for $\Delta(\Pi_n^*)/S_\lambda$, we must use block content as well as size to determine block equivalence, and we use our chosen block order that satisfies the lengthening condition to decide which offspring blocks are left children and which are right children and also how to sort blocks, but otherwise the proof will be identical to that in [He, p. 14-24], by virtue of the properties of the lengthening discussion, to be discussed next.

We claim that the lengthening condition ensures that we may replace any pair of consecutive bar insertions which proceed either (1) from right to left bar or (2) which insert bars from left to right in a single block creating left children decreasing in size from left to right, by a lexicographically earlier saturated chain which overlaps ours in a codimension one face, yielding a topological descent in our saturated chain at the rank in between the two bar insertions. In case 1, if the two refinement steps refine distinct blocks from right to left, then it is clear that these may also be refined left to right. The other possibility for case 1 is that consecutive refinement steps split a single block B into children B_1, B_2, B_3 by first refining B into children B_1B_2, B_3, and that $B_1B_2 < B_3$, $B_1 < B_2$, so that bar insertions proceed right to left; then the lengthening condition asserts that $B_1 < B_1B_2$, so that the refinement first to children B_1, B_2B_3 gives a lexicographically smaller chain. In case 2, the first step refines a block B into children B_1, B_2B_3 and the next step refines B_2B_3 into children B_2, B_3. We have that $B_1 > B_2$ and $B_2 < B_3$, which implies that $B_1 > \min(B_2, B_1B_3)$. We get a lexicographically smaller chain by first splitting B instead into children B_2, B_1B_3. These implications of the lengthening condition are adopted from [HK]. □

Now let us use the partitioning for $\Delta(\Pi_n)/S_1 \times S_{n-1}$ derived from the distinguished block order to obtain the following result.

Theorem 4.2. *If $S = \{1, \dots, i\}$, then $b'_S(n) = 1$ and otherwise $b'_S(n) > 1$.*

Proof. Let $\{s, t, \dots, t\}$ denote the set of objects to be partitioned. Notice that one may construct a saturated chain in $\Delta(\Pi_n^*)/S_{n-1} \times S_1$ achieving any desired collection S^* of topological descents by inserting bars in the ordered set $s, t, \dots, t$ as follows: for each maximal (possibly empty) string of ascents followed by a descent, we place bars left to right filling the rightmost collection of available spots. Finally,

we insert bars left to right for the terminal string of ascents, if there is one. In this fashion, we achieve any S, implying $b'_S(n) \geq 1$ for all S. For example, $S^* = \{1, 4, 5\}$ is achieved in $\Delta(\Pi_8^*)/S_1 \times S_7$ by $s|_6t|_7t|_5t|_2t|_3t|_4t|_1t$ (letting subscripts denote ranks of bar insertions) since this gives $w(S^*) = DA^2D^2A$.

Next we show $b'_{1,\ldots,i}(n) = 1$. Consider a word $w(S^*) = A^{n-i-2}D^i$, namely the case of $S = \{1, \ldots i\}$. A bar insertion isolating the s followed by any other bar insertion must comprise an ascent, while any bar insertion immediately before one isolating the unique s must be a descent. Thus, for $S = 1, \ldots, i$, the s must be isolated in either the first or the last refinement step. It cannot be the first step since $b_{1,\ldots,j}(n) = 0$, which means it would be impossible to refine the remaining nontrivial block of $n-1$ identical letters achieving a word $A^{n-j-3}D^i$. Thus, the step splitting off the s must come last. Until the first descent, bars must be inserted left to right creating blocks of nondecreasing size (after the first block which is automatically smallest by virtue of containing the s). The rightmost of these newly created blocks must have size 1, to avoid having a later ascent at any point after the first descent. Since the rightmost block has size one, these increasing blocks all must have size one. Thus, only the block containing s may be nontrivial after the initial series of consecutive ascents, so we must begin by inserting bars from left to right distance one apart filling up the rightmost available set of positions. Now to avoid further ascents, we have no choice but to proceed right to left refining the block containing s. Since there is only one such saturated chain, we conclude that $b'_{\{1,\ldots,i\}}(n) = 1$.

For $S \neq \{1, \ldots, i\}$, we obtain a facet achieving S as in the first paragraph, but the fact that we have a descent immediately before an ascent, gives enough flexibility to guarantee an alternative facet also achieving S, constructed as follows: the bar for the descent immediately preceding a string of ascents may be placed one position farther to left than the above greedy algorithm would choose. If the string of ascents is followed by a descent, then we put a bar at the rightmost position that is still vacant when we encounter the descent; if the string of ascents concludes the entire string, then we may place the bar insertion for the last ascent into this rightmost position. In any case, $b'_S(n) > 1$ for $S \neq \{1, \ldots, i\}$. □

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